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Renormalization group: critical surface of arbitrary co-dimension

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Abstract. The renormalization group is employed to analyse a model Hamiltonian which gives arbitrary odd-integral values for the critical exponent δ (where $H \sim M^{\delta}$). Correspondingly, the co-dimension of the critical surface and the number of 'relevant' variables are each equal to $\frac{1}{2}(\delta+1)$.

1. Introduction

The renormalization group (Wilson 1971) has proved a useful tool in the theory of critical phenomena, contributing both to theoretical understanding and to numerical results. The models which are susceptible to more or less explicit calculation include the Gaussian and spherical models (eg Ma 1973), while the ϵ expansion (Wilson and Fisher 1972) provides an asymptotic series useful for dimension close to four. Wilson and Kogut (1974) give an extensive review of the subject. Certain scaling properties, justified by physical arguments in Wilson's (1971) theory, are satisfied identically in a model of Baker (1972).

In the usual physical situation the number of variables which must be fixed to obtain the critical point is two (temperature and magnetic field in the case of a ferromagnet). We say that the critical surface has co-dimension two, and T and H are the only two 'relevant' variables in the sense of Wilson (1971); ie, they correspond to the (only) two eigenvalues of the linearized form of the renormalization group transformation R_s which exceed unity. In this paper we discuss a class of models for which such a critical surface of co-dimension two exists, but, in addition, there is a sequence of critical surfaces of higher co-dimension, each contained in the closure of the previous one. For these smaller critical surfaces not only must T and H have their critical values, but certain relations among the parameters in the Hamiltonian must be satisfied. The critical surface of co-dimension two corresponds to critical exponents (Fisher 1967) of the usual mean field type; in particular, $\delta = 3$. Higher odd-integral powers of δ are represented by critical surfaces of higher co-dimension $\frac{1}{2}(\delta+1)$. The critical exponent γ is equal to unity and most of the usual scaling laws (Fisher 1967) hold, at least below the critical temperature.

The model is essentially that of McKerrell and Bowers (1972, referred to in the following as MB) and is rather unrealistic physically. Its great advantage is the ease with which explicit calculations can be performed, either by using conventional statistical mechanical methods, as in MB, or, as here, by employing the renormalization group.

Our purpose, therefore, is not to add one more to the (regrettably small) number of exactly soluble models in the field, but rather to contribute to the understanding of the renormalization group technique by demonstrating its application to a model in which critical surfaces of arbitrarily high co-dimension appear and can be treated explicitly.

For the renormalization group we follow, to a large extent, the methods and notation of Ma's (1973) very clear introduction to the subject.

2. The model

The model (MB) is of a cooperative assembly of N 'spins' $\phi(x_j) = \pm 1$ which interact by two-point, four-point, ... potentials according to the Hamiltonian

$$\mathscr{H}_a = -J \sum_{t=1}^{\infty} a_{2t} N^{1-2t} \sum_{j_1 \ldots j_{2t}} \phi(x_{j_1}) \ldots \phi(x_{j_{2t}}) - Hm \sum_j \phi(x_j) \tag{1}$$

where m, the magnetic moment per spin, and J are positive constants and

$$a = (a_2, a_4, \ldots) \tag{2}$$

parametrizes the model. We may regard the spin $\phi(x_j)$ as situated at a point x_j of a lattice in a space of dimension d, but the physically unrealistic feature of this Hamiltonian is, of course, that the potentials are independent of the distances between the points x_j. This is also its great simplifying feature since if, following Ma, we define the Fourier components ϕ_k of the spin field by

$$\phi_k = N^{-1/2} \sum_j e^{-ik_j x_j} \phi(x_j), \tag{3}$$

then $\mathscr{H}_a$ depends only on ϕ_0:

$$\mathscr{H}_a = -J \sum_{t=1}^{\infty} a_{2t} N^{1-t} \phi_0^{2t} - HmN^{1/2}\phi_0. \tag{4}$$

We define the partition function as usual by

$$Z_N(T, H, a) = 2^{-N} \sum \exp(-\mathscr{H}_a/kT) \tag{5a}$$

$$= 2^{-N} \sum_{\phi_0} \Omega(\phi_0) \exp(-\mathscr{H}_a/kT). \tag{5b}$$

In (5a) we use (1) for $\mathscr{H}_a$ and the sum is over all configurations of the $\phi(x_j) = \pm 1$; in (5b) $\mathscr{H}_a$ is replaced by its expression in terms of ϕ_0, given in (4), the sum is over the possible values of ϕ_0, ie, according to (3), from $-N^{1/2}$ to $+N^{1/2}$ in steps of $2N^{-1/2}$, and $\Omega(\phi_0)$ is the number of configurations of the $\phi(x_j)$ which lead to the particular value ϕ_0. If we let r denote the number of $\phi(x_j)$ which are -1, say, then

$$\Omega(\phi_0) = \binom{N}{r}, \qquad \phi_0 = N^{-1/2}(N - 2r). \tag{6}$$

The Gibbs free energy of the model (we choose the notation of Stanley 1971) is

$$G(T, H, b) = -\lim_{N\to\infty} kTN^{-1} \ln Z_N(T, H, a) \tag{7a}$$

$$= -\lim_{N\to\infty} kTN^{-1} \ln \int_{-\infty}^{\infty} d\phi_0 \exp\left(\sum_{t=1}^{\infty} b_{2t} N^{1-t} \phi_0^{2t} + HmN^{1/2}\phi_0/kT\right) \tag{7b}$$

where

$$b_{2t} = Ja_{2t}/kT - [2t(2t-1)]^{-1}. \tag{8}$$

In going from (7*a*) to (7*b*) we have used Stirling's asymptotic formula for the factorials in $\binom{N}{r}$ and expanded the resulting logarithms in series. The replacement of the sum (over values of ϕ_0) by an integral is justified in the thermodynamic limit ($N \to \infty$), provided that we restrict attention to values of b (and hence a) for which the integral converges. Sufficient generality for our purposes is retained on assuming, as in MB, that only a finite number of the a_{2t} are nonzero.

We denote by μ a general point of the parameter space $\mathcal{M}$, where we include the external physical variables T, H and the parameters a_{2t} of the Hamiltonian. Thus two alternative representations of μ in terms of coordinates in $\mathcal{M}$ are

$$\mu = (T, H, a) = (T, H, a_2, a_4, \ldots) \tag{9a}$$

$$= (T, H, b) = (T, H, b_2, b_4, \ldots). \tag{9b}$$

The renormalization group transformation R_s acts on $\mathcal{M}$:

$$R_s\mu = \mu' \tag{10}$$

and is realized as follows. In the general case (Ma 1973) where the Hamiltonian depends on all the ϕ_k with $|k|$ less than some cut-off Λ, those ϕ_k with $|k| > \Lambda/s$ are integrated out and the maximum wavevector is restored to Λ by the transformation

$$\phi_k = s^{1-\eta/2}\phi'_{sk} \tag{11}$$

where η is a parameter to be chosen later. In the present model only ϕ_0 appears and we write

$$\phi_0 = s^{1-\eta/2}\phi'_0. \tag{12}$$

However, the multiplication of wavevectors by s, indicated in (11), means that the density of points in k space is reduced by a factor s^{-d} and, correspondingly, that the number of spins described is reduced:

$$N = s^d N'. \tag{13}$$

If the substitutions (12) and (13) are made in (7*b*), the expression in $(\ldots)$ can be restored to its original form, but in terms of primed variables b'_{2t}, N', ϕ'_0, H', by defining

$$b'_{2t} = b_{2t}s^{-d(t-1)+t(2-\eta)} \tag{14a}$$

$$H' = Hs^{(d+2-\eta)/2}. \tag{14b}$$

This leads to the relation

$$G(T, H, b) = s^{-d}G(T, H', b') \tag{15}$$

which will form the basis of most of the subsequent discussion. We should like to emphasize that the deduction of (15) from (7*b*) and (14) is independent of the semi-physical argument used above.

Equations (14), together with the fact that T is unchanged, supply the explicit realization of the renormalization group transformation (10) in the present model.

The renormalization group method proceeds by investigating any fixed points of the transformation R_s towards which the system tends with increasing s, ie points μ^* of $\mathscr{M}$ such that

$$R_s\mu^* = \mu^*, \qquad \mu' = R_s\mu \to \mu^* \qquad \text{as } s \to \infty. \tag{16}$$

This leads to the critical point(s) of the model while the behaviour of equation (15) near the fixed point leads to the properties of the model near the critical point and, in particular, to values for the critical exponents.

At a fixed point (T^*, H^*, b^*) only one component of b^*, say $b^*_{2t_0}$, can be nonzero, as we see from (14*a*), which leads also to the result

$$(2-\eta)/d = (t_0-1)/t_0. \tag{17}$$

It is clear that $\eta \leqslant 2$ (the case $\eta = 2, t_0 = 1$ does not lead to critical point behaviour and will be excluded in the following) and so it follows from (14*b*) that a fixed point has $H^* = 0$.

We also define

$$\delta = 2t_0 - 1 = \frac{d+2-\eta}{d-2+\eta} \tag{18}$$

an odd integer which will turn out to be the usual critical exponent. For this reason we will use δ, rather than t_0, to label the transformation $R_s = R_s^{(\delta)}$ and the fixed point $\mu^* = \mu^{(\delta)}$ which we have just found. Equation (14*a*) becomes

$$b'_{2t} = b_{2t}s^{d(\delta+1-2t)/(\delta+1)}. \tag{19}$$

It is now clear that, as $s \to \infty$,

$$b'_{2t} \to b^{(\delta)}_{2t} = 0 \qquad \text{for } 2t > \delta+1 \tag{20a}$$

$$b'_{2t} = b^{(\delta)}_{2t} = b_{2t} \qquad \text{for } 2t = \delta+1 \tag{20b}$$

while, in order to ensure that

$$b'_{2t} \to b^{(\delta)}_{2t} = 0 \qquad \text{for } 2t < \delta+1, \tag{20c}$$

we must impose the $(\delta-1)/2$ conditions

$$0 = b_{2t} \equiv Ja_{2t}/kT - [2t(2t-1)]^{-1}; \qquad 2t < \delta+1. \tag{21}$$

First let us consider the case $\delta = 3$. Then (21) yields a single condition which may be satisfied by fixing the temperature at a particular value T_c given in terms of the parameters J and a_2 of the Hamiltonian by

$$T_c = 2Ja_2/k. \tag{22}$$

This is the generic case: fixing $H = 0$, $T = T_c$ ensures that μ lies on the critical surface $\mathscr{S}^{(3)} \subset \mathscr{M}$ (provided that $b_4 \neq 0$: see below), where we define

$$\mathscr{S}^{(\delta)} = \{\mu : R_s^{(\delta)}\mu \to \mu^{(\delta)} \text{ as } s \to \infty\}. \tag{23}$$

This generic critical surface $\mathscr{S}^{(3)}$ has co-dimension two in $\mathscr{M}$.

The situation with higher values of δ is rather different. Choosing $T = T_c$ enables us to satisfy only one of the conditions (21). The others must be interpreted as restrictions on the coefficients a_{2t} in the Hamiltonian, namely that the a_{2t} appearing in equation (21) must be proportional to $[2t(2t-1)]^{-1}$. (These conditions were imposed *a* *priori* in MB.) This means that the fixed points $\mu^{(\delta)}$ for $\delta > 3$ are non-generic. They are realizable mathematically but a slight perturbation of the Hamiltonian (a slight change in a_4) means that b_4 is no longer zero at the critical temperature and $R_s^{(\delta)}b_4$ diverges as $s \to \infty$. The transformation with a non-trivial fixed point is then $R_s^{(3)}$, leading, as we shall see, to the usual mean field critical exponents.

3. The critical exponents

We turn now to the calculation of the critical exponents corresponding to the fixed point $\mu^{(\delta)}$ of $R_s^{(\delta)}$. (For definitions of the critical exponents see Fisher (1967); our methods are similar to those of Ma (1973) but we choose a different starting point in the Gibbs free energy $G(T, H, b)$.) When referring to $\mu^{(\delta)}$ and its properties we shall assume that the requisite conditions (21) on the parameters a_{2t} of the Hamiltonian are satisfied.

From (15) and (14*b*) we have

$$G(T, H, b) = s^{-d}G(T, Hs^{(d+2-\eta)/2}, b'). \tag{24}$$

The corresponding relation for the magnetization is obtained by differentiation with respect to H:

$$\begin{aligned} M(T, H, b) &= -G_2(T, H, b) = -s^{-d}s^{(d+2-\eta)/2}G_2(T, H', b') \\ &= s^{-(d-2+\eta)/2}M(T, H', b') \end{aligned} \tag{25}$$

where the subscript 2 denotes differentiation with respect to the second variable.

Now we know that for $T = T_c$ the parameters b' approach their fixed-point values $b^{(\delta)}$ as $s \to \infty$, independently of the value of H. We choose s (so far arbitrary) to be the following function of H:

$$s = |H|^{-2/(d+2-\eta)} \tag{26}$$

so that $s \to \infty$ as $H \to 0\pm$ in such a way that H' is constant (and equal to the sign of H). Thus it is reasonable to suppose (and this is the usual assumption of the renormalization group method) that $M(T_c, H', b')$ tends to a nonzero constant $\pm c_1$, say, and we have

$$M(T_c, H, b) \sim c_1 H^{(d-2+\eta)/(d+2-\eta)} = c_1 H^{1/\delta}, \qquad H \to 0\pm, \tag{27}$$

showing that δ is indeed the usual critical exponent.

The other critical exponents defined for the present model refer to the behaviour for T near T_c and H equal to zero. We look first at the case $T < T_c$. We see from (19) that the dominant singular component of b' is

$$b'_2 = b_2 s^{d(\delta-1)/(\delta+1)} = (T_c - T)s^{2-\eta}/2T \tag{28}$$

where we have used (8), (18) and (22).

We can arrange this to be of order unity by choosing

$$s = (T_c - T)^{-1/(2-\eta)}; \tag{29}$$

then $s \to \infty$ as $T \to T_c -$. Thus we may suppose that $G(T, 0, b')$ will be of order unity. (For $H = 0$ we see from (7*b*) that the explicit T dependence is just the factor T in front: the rest comes from the T dependence in b.) Thus we have, from (15),

$$G(T, 0, b) \sim c_2(T_c - T)^{d/(2-\eta)} = c_2(T_c - T)^{(\delta+1)/(\delta-1)}, \tag{30}$$

showing that the critical exponent α' is given by

$$2-\alpha' = (\delta+1)/(\delta-1), \qquad \alpha' = (\delta-3)/(\delta-1). \tag{31}$$

A similar calculation applied to (25) shows that

$$M(T,0,b) \sim c_3(T_c-T)^{(d-2+\eta)/2(2-\eta)} = c_3(T_c-T)^{1/(\delta-1)}. \tag{32}$$

Thus

$$\beta = 1/(\delta-1). \tag{33}$$

The corresponding result for the susceptibility $\chi = \partial M/\partial H$ is, from (25),

$$\chi(T,0,b) = s^{-(d-2+\eta)/2}s^{(d+2-\eta)/2}\chi(T,0,b') \sim c_4(T_c-T)^{-1}, \tag{34}$$

so we have

$$\gamma' = 1. \tag{35}$$

Each successive differentiation of (25) with respect to H leads to an additional factor

$$s^{(d+2-\eta)/2} = (T_c-T)^{-(d+2-\eta)/2(2-\eta)} \tag{36}$$

so that we obtain for the gap exponent

$$\Delta' = \tfrac{1}{2}(d+2-\eta)/(2-\eta) = \delta/(\delta-1). \tag{37}$$

The critical exponents which we have calculated satisfy the scaling laws

$$2-\alpha'-\beta = \beta+\gamma' = \beta\delta = \Delta'. \tag{38}$$

The definition (18) also has the form of a scaling law, but our model does not allow the interpretation of η as a critical exponent since the correlation function has no space dependence.

It might appear at first sight that the above results for $T \to T_c-$ should hold also for $T \to T_c+$, but only the argument for χ goes through unchanged; thus

$$\gamma = \gamma' = 1. \tag{39}$$

Of course, $M = 0$ when $H = 0$ for $T > T_c$ as usual, but in our model

$$G(T,0,b) = 0, \qquad T > T_c, \tag{40}$$

so α is not defined. This result and that given in MB on the non-uniqueness of $\Delta(T > T_c)$ require more detailed investigation of the precise form of G than is natural in the renormalization group approach. Our remarks above concerning quantities being of order unity are oversimplifications of the situation for $T > T_c$ (thus great care is necessary in applying renormalization group techniques).

For $\delta = 3$ only, the gap exponent Δ is well defined for $T > T_c$. For the generic fixed point $\mu^{(3)}$ we have

$$\alpha' = 0, \qquad \beta = \tfrac{1}{2}, \qquad \gamma = \gamma' = 1, \qquad \delta = 3, \qquad \Delta = \Delta' = \tfrac{3}{2}, \tag{41}$$

the usual mean field critical exponents.

We conclude with some remarks on the relation between the various critical surfaces $\mathscr{S}^{(\delta)}$ in $\mathscr{M}$. Each satisfies $H = 0$, $T = T_c$, where T_c is defined by $b_2(T_c) = 0$. Each successive value of δ, after $\delta = 3$, requires an additional component of b to vanish at T_c, imposing an extra restriction on a as discussed earlier. On $\mathscr{S}^{(5)}$, $b_4(T_c) = 0$; on $\mathscr{S}^{(7)}$, in addition $b_6(T_c) = 0$, and so on. The lowest critical surface $\mathscr{S}^{(3)}$ is of co-dimension two in $\mathscr{M}$ and its closure contains all higher $\mathscr{S}^{(\delta)}$; $\mathscr{S}^{(\delta)}$ has co-dimension $\frac{1}{2}(\delta+1)$ and its closure contains all $\mathscr{S}^{(\delta')}$ for $\delta' > \delta$.

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